

Math 241
Summer 2018
Exam 1 - Practice
7/13/15
Time Limit: 50 Minutes

Name (Print):

Solutions

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	15	
7	15	
8	15	
Total:	145	

1. (20 points) Find the following limits:

$$\text{a) } \lim_{x \rightarrow 2} \frac{(x^2 - 4)(x + 2)}{(x^2 - 5x + 6)(x + 1)}$$

$$= \lim_{x \rightarrow 2} \frac{(x-2)(x+2)(x+2)}{(x-2)(x-3)(x+1)}$$

$$= \lim_{x \rightarrow 2} \frac{(x+2)^2}{(x+3)(x+1)} = \frac{(2+2)^2}{(2+3)(2+1)} = \frac{16}{15}$$

$$\text{b) } \lim_{x \rightarrow 0} \frac{\tan(2x)}{3x} = \lim_{x \rightarrow 0} \frac{\left(\frac{\sin(2x)}{\cos(2x)} \right)}{3x} = \lim_{x \rightarrow 0} \frac{\sin(2x)}{3x} \cdot \frac{1}{\cos(2x)}$$

$$= \frac{1}{3} \lim_{x \rightarrow 0} \frac{2}{2} \frac{\sin(2x)}{x} \cdot \frac{1}{\cos(2x)}$$

$$= \frac{2}{3} \lim_{x \rightarrow 0} \frac{\sin(2x)}{2x} \cdot \frac{1}{\cos(2x)}$$

$$= \frac{2}{3} \cdot 1 \cdot 1$$

$$= \frac{2}{3}$$

$$\text{c) } \lim_{x \rightarrow 1} \frac{\sqrt{x+8} - 3}{x - 1}$$

$$= \lim_{x \rightarrow 1} \frac{(\sqrt{x+8} - 3)}{(x - 1)} \frac{(\sqrt{x+8} + 3)}{(\sqrt{x+8} + 3)} = \lim_{x \rightarrow 1} \frac{x+8-9}{(x-1)(\sqrt{x+8} + 3)}$$

$$= \lim_{x \rightarrow 1} \frac{1}{\sqrt{x+8} + 3}$$

$$= \frac{1}{\sqrt{9} + 3} = \frac{1}{6}$$

$$\text{d) } \lim_{x \rightarrow 1^-} \frac{\sqrt{3x}|x-1|}{(x-1)}$$

Because $x \rightarrow 1^-$, $x-1 < 0$ and whence $|x-1| = -(x-1)$.

$$\lim_{x \rightarrow 1^-} \frac{\sqrt{3x}(-(x-1))}{(x-1)} = \lim_{x \rightarrow 1^-} -\sqrt{3x} = -\sqrt{3}$$

2. (20 points) Find the following limits:

$$\begin{aligned}
 \text{a) } \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 + 1} + \sqrt{x^2 + x}} &= \lim_{x \rightarrow -\infty} \frac{x}{|x|(\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 + \frac{1}{x}})} \\
 &= \lim_{x \rightarrow -\infty} \frac{-1}{\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 + \frac{1}{x}}} \\
 &= \frac{-1}{\sqrt{1+0} + \sqrt{1+0}} = \frac{-1}{2}
 \end{aligned}$$

$x < 0, \text{ so, } |x| = -x$

b) $\lim_{x \rightarrow \infty} \frac{2 \sin(x)}{x^2 + 1}$ Consider the inequality

$-1 \leq \sin(x) \leq 1$, it gives $-2 \leq 2 \sin(x) \leq 2$ and further,
 that $\frac{-2}{x^2 + 1} \leq \frac{2 \sin(x)}{x^2 + 1} \leq \frac{2}{x^2 + 1}$.

Now, because $\lim_{x \rightarrow \infty} \frac{-2}{x^2 + 1} = 0 = \lim_{x \rightarrow \infty} \frac{2}{x^2 + 1}$, $\lim_{x \rightarrow \infty} \frac{2 \sin(x)}{x^2 + 1} = 0$ by S.W.T.

c) $\lim_{x \rightarrow 2^-} \frac{x^2 - 10}{x^2 - 4} = +\infty$

$$\begin{aligned}
 \text{d) } \lim_{x \rightarrow \infty} \frac{(x^{5/3} - 10) \sqrt[3]{x}}{2x^2 + x - 4} &= \lim_{x \rightarrow \infty} \frac{x^{2/3} - 10x^{1/3}}{2x^2 + x - 4} \\
 &= \lim_{x \rightarrow \infty} \frac{x^2(1 - \frac{10}{x^{5/3}})}{x^2(2 + \frac{1}{x} - \frac{4}{x^2})} \\
 &= \frac{1 - 0}{2 + 0 - 0} \\
 &= \frac{1}{2}
 \end{aligned}$$

3. (20 points) a) State the Intermediate Value Theorem.

Given a continuous function, f , on a closed interval $[a,b]$, and a number, α , in between $f(a)$ and $f(b)$, there will always exist a number, x_0 , with $a \leq x_0 \leq b$ such that

$$f(x_0) = \alpha !$$

- b) Use the Intermediate Value Theorem to show that the equation $\sin(x) + x = 1$ has a solution.

Let $f(x) = \sin(x) + x - 1$, and note that this function is continuous.

As $f(0) = -1$ and $f(2\pi) = 2\pi - 1$ (which is positive), the IRT gives us an $x_0 \in [0, 2\pi]$ such that

$$f(x_0) = 0.$$

This gives $\sin(x_0) + x_0 - 1 = 0$ and, so,

$$\sin(x_0) + x_0 = 1.$$

Whence, x_0 is a solution to the equation.

4. (20 points) Use the **definition** of the derivative as a limit to find $f'(x)$ if $f(x) = \frac{1}{x+1}$.
 Note: using derivative rules will get no points!

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x+1 - (x+h+1)}{(x+h+1)(x+1) \cdot h} \\
 &= \lim_{h \rightarrow 0} \frac{-h}{(x+h+1)(x+1) \cdot h} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{(x+h+1)(x+1)} \\
 &= \frac{-1}{(x+1)^2}
 \end{aligned}$$

Note:

In this type of problem, it is wise to check your answer:

$$\begin{aligned}
 \frac{d}{dx}(x+1^{-1}) &= -1(x+1)^{-2} \cdot 1 \text{ by chain rule,} \\
 &= \frac{-1}{(x+1)^2} \text{ (First power change)}
 \end{aligned}$$

5. Find the following derivatives: (warning! no partial credit)

(a) (5 points) $f(x) = \sqrt{x} + 7x + \frac{1}{x} = x^{1/2} + 7x + x^{-1}$

$$f'(x) = \frac{1}{2\sqrt{x}} + 7 + \frac{-1}{x^2}$$

Note: $\frac{d}{dx} \left(\frac{1}{x} \right) \neq \frac{1}{1}$

(b) (5 points) $g(x) = \tan(x)\sqrt{x^2 + 1}$

$$g'(x) = \sec^2(x)\sqrt{x^2+1} + \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x \tan(x)$$

(c) (5 points) $h(x) = \frac{3\sec(2x)}{\frac{1}{x^2} + x} = \frac{3\sec(2x)}{x^{-2} + x}$

$$h'(x) = \frac{3\sec(2x)\tan(2x) \cdot 2(x^{-2} + x) - (-2x^{-3} + 1)3\sec(2x)}{(x^{-2} + x)^2}$$

(d) (5 points) $k(x) = \sin(1 + \sqrt{x^2 + 2})$

$$k'(x) = \cos(1 + \sqrt{x^2 + 2}) \cdot \frac{1}{2}(x^2 + 2)^{-1/2} \cdot 2x$$

6. (15 points) Given $x^2 + y^3 = xy^2$, find $\frac{dy}{dx}$. note: Be prepared to give back the equations of the normal and tangent lines at a particular point on the actual exam.

$$\frac{d}{dx}(x^2 + y^3) = \frac{d}{dx}(xy^2)$$

$$\Leftrightarrow 2x + 3y^2 \cdot \frac{dy}{dx} = y^2 + x \cdot 2y \cdot \frac{dy}{dx}$$

$$\Leftrightarrow 3y^2 \frac{dy}{dx} - 2xy \frac{dy}{dx} = y^2 - 2x$$

$$\Leftrightarrow \frac{dy}{dx} = \frac{y^2 - 2x}{3y^2 - 2xy}$$

7. (15 points) Given the position function $p(t) = \sin(t) + t^2 + t + 2$ (in feet where t is in seconds), find the velocity function $v(t)$, and the acceleration function $a(t)$. What is the object's initial velocity (a.k.a. $v'(0)$)? What is the object's acceleration at $t = 2\pi$?

Typo: Initial Velocity is $v(0)$, not $v'(0)$.
($v'(0)$ would be initial acceleration)

$$V(t) = P'(t) = \cos(t) + 2t + 1$$

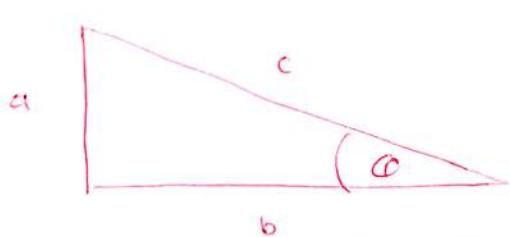
$$a(t) = V'(t) = -\sin(t) + 2$$

Initial Velocity: $V(0) = \cos(0) + 2(0) + 1$
 $= 2$

$$a(2\pi) = -\sin(2\pi) + 2
= 2$$

8. (15 points) Batman is particular about kites. He ONLY flies a kite at a height of exactly 300ft., always. Today is no different. Also, today, the wind pushes the kite and it moves horizontally in the air at a rate $25 \frac{\text{ft}}{\text{sec}}$.

- a) How fast must he be letting out string for the kite to remain at the constant height of 300ft. at the precise moment he has let 500 ft. of string out?



$$\underline{\text{Given:}} \quad \frac{db}{dt} = 25 \frac{\text{ft}}{\text{sec}}$$

$$\underline{\text{Want:}} \quad \frac{dc}{dt}$$

Note: At this time $b = 400$.

$$c^2 = a^2 + b^2$$

$$\frac{d}{dt}(c^2) = \frac{d}{dt}(a^2 + b^2)$$

$$2c \cdot \frac{dc}{dt} = 2a \frac{da}{dt} + 2b \frac{db}{dt}$$

$$a \text{ is constant so, } \frac{da}{dt} = 0$$

$$\text{Now, } \frac{dc}{dt} = \frac{b \cdot \frac{db}{dt}}{c}$$

$$= \frac{400 \cdot 25}{500}$$

$$= \frac{4 \cdot 25}{5} = 4 \cdot 5 = 20 \frac{\text{ft}}{\text{sec.}}$$

- b) How fast is the angle between the string and the ground changing at this time?

(see picture above) want: $\frac{d\phi}{dt}$

$$\tan(\phi) = \frac{a}{b} \quad \text{and side } a \text{ is constant length, so,}$$

$$\tan(\phi) = 300 \cdot b^{-1}$$

$$\text{(after } \frac{d}{dt} \text{ we get)} \quad \sec^2(\phi) \cdot \frac{d\phi}{dt} = \frac{-300}{b^2} \cdot \frac{db}{dt}$$

$$= \frac{-300}{(400)^2} \cdot 25$$

$$\cos(\phi) = \frac{b}{c}, \quad \text{and } \sec(\phi) = \frac{5}{4},$$

$$\begin{aligned} &= \frac{400}{500} \\ &= \frac{4}{5} \end{aligned}$$

$$\text{so, } \frac{d\phi}{dt} = \frac{-3}{4 \cdot 400} \cdot 25 \cdot \frac{4^2}{5^2} = \frac{-12}{400} = \frac{-3}{100} \frac{\text{rad}}{\text{sec.}}$$